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Relativity From Quantum Theory[1]

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Starting from the fact that the geodesic structure of the projective space of quantum pure states gives a natural explanation for the fundamentality of spin 1/2 systems in relativistic quantum theory, and making use of the inducing construction for infinite-dimensional representations of groups in vector bundles over projective space, a proposal for unifying physical theory in terms of a possible derivation of relativistic physics from pure quantum theory is presented.

In the spirit of this conference as set by the opening paper of P.A.M. Dirac—the investigation of pretty mathematics as a source of explanations for physical phenomena—the present paper is a progress report on an exploration of the mathematics of the projective geometry underlying quantum theory as a possible explanation for the appearance of relativity in physics. What originally stimulated interest in this line of investigation was noticing, from some purely mathematical computations the author was making in the summer of 1980, that the geodesics of an arbitrary-dimensional projective geometry (the ray space of pure states in quantum theory) corresponded exactly to the evolution of spin 1/2 systems in quantum physics, in the following sense: if one requires that a quantum system evolve along the simplest possible trajectory—a geodesic of quantum ray space—then the motion of the corresponding state vector is limited exactly to a two-dimensional subspace of the ambient Hilbert space from which the ray space is derived. In hindsight this result is obvious, since projective space is essentially a locally spherical manifold [technically, a space of constant holomorphic sectional curvature (Kobayashi and Nomizu, 1969) and so its geodesics must look like great circles, i.e., two-dimensional plane curves.

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Another way of saying this is that the geodesics of n-dimensional real or complex projective space ($P_n R$ or $P_n C$) are embedded copies of real one-dimensional projective space $P_1 R$.

Now Dirac has consistently called attention to the rather surprising nature of the fact that spin 1/2 is simpler than any other spin value, for example in this treatment of electron spin in *The Principles of Quantum Mechanics* and, more recently, in describing at a conference similar to this one how the search for pretty mathematics contributed to his discovery of the equation that bears his name:

> Surprisingly, the particle with spin 1/2 appeared as simpler to treat than the particle with no spin. One would have expected that one would first have to solve the problem of the particle with no spin and then subsequently bring in the spin. But the mathematics showed otherwise. The mathematics led the way. (Dirac, 1978)

Up to the present time, of course, the physical explanation of the fundamentality of spin 1/2 has been that originally given by Dirac: impose restrictions from classical relativity theory on quantum theory and spin 1/2 comes out as a property of the basic solutions of the resulting relativistically invariant Dirac equation. In the geodesic structure of the projective space of quantum states, however, we seem to have an alternate, purely geometric, explanation without the need to impose relativity on quantum theory from the outside. The fundamentality of the spin-1/2 case in quantum theory follows naturally from the fundamentality of geodesics in any manifold as the simplest possible curves in the manifold. If this piece of what is standardly regarded as relativistic quantum physics can be derived from fundamental quantum theory alone without benefit of relativity theory, the question naturally arises as to how much more of relativistic physics (and the rest of modern physics) might come out of the basic nonlinear projective space geometry of quantum states if it is taken seriously as the fundamental geometry of the observable universe and the methods of modern differential geometry are applied to develop the consequences. That such methods have not previously been used in the way proposed here seems mainly due to the almost exclusive emphasis in the development of quantum theory on the homogeneous coordinatizations available over projective space (standard state vector or wave function representation of states); it is the use of such coordinatizations that is responsible for the linear appearance of quantum theory and for the fact that properly nonlinear differential geometric methods seem superfluous. An example of just how much can be concealed by homogeneous coordinatizations was pointed out early in the history of quantum theory by Hermann Weyl (1931) under the title "Quantum Kinematics as an Abelian Group of Rotations." He noted that an Abelian group of transformations of quantum ray space was induced by the non-

Abelian action on the ambient Hilbert space of operators $e^{ixq}, e^{iyp}, x, y \in R, qp - pq \sim I$. If the fundamental commutation relations of standard quantum theory are an effect of homogeneous coordinatizations rather than an intrinsic property of the transformations of states, how much more might be hidden in the interrelation between projective geometry and its linear space representations? It does not take much investigation of the matter to lead to the suspicion that most, if not all, of the infinities appearing in relativistic quantum physics might well be the result of ignoring chart limitations—a mathematically hazardous enterprise over any manifold, as general relativists learned some years ago. But if one is unaware of working over a nonlinear manifold, it is quite easy to encounter such predictions as an infinite amount of land just north of the immense continent of Greenland.

In any case, the currently available results of an ongoing investigation of such matters are presented here. As will quickly become evident, the results at present are more in the nature of tantalizing leads and directions for more work, rather than final answers in definitive form. Enough has been uncovered, however, to lead to the conviction that the work should be carried to full conclusion. If successful, the reward will be the full unification of 20th century physics in terms of some very simple and pretty mathematics.

In light of the questions posed above, a purely mathematical result concerning the natural structures available over projective space takes on new significance. The construction has previously been applied for getting infinite-dimensional representations of groups, and we paraphrase slightly the relevant facts as given in that context by Robert Hermann (1966) in *Lie Groups for Physicists* [cf. also Hermann's *Topics in the Mathematics of Quantum Mechanics* (1977) for the general formulation of quantum theory in terms of projective geometry]:

> For each integer $r \geqslant 0$, there is a homogeneous line bundle on $P_n(C)$ with an $SL(n+1, C)$-invariant unitary inner product on the cross sections. The action of $SU(n+1)$ on the cross sections defines a reducible unitary representation of $SU(n+1)$ that decomposes into traceless symmetric tensors of the type $A^{j_1 \cdots j_k}_{i_1 \cdots i_{k+r}}$, and each representation occurs only once. As k runs from 0 to ∞ the representations form a "ladder." The unitary representation of $SL(n+1, C)$ on the cross sections is irreducible, and the noncompact operators in the Lie algebra of $SL(n+1, C)$ shift up and down the ladder.

Before going into the computational details of this construction, we look at some of the applications suggested by the lowest orders. Since the result is basically derived from the mathematics of induced representations, we will for brevity from now on refer to it as the *inducing construction*.

For $n = 1$ the construction gives an infinite-dimensional Hilbert space H_∞ of cross sections of a complex line bundle over P_1C, equipped with a natural unitary representation of $SL(2, C)$ on the cross sections $\psi \in H_\infty$. Now of course the wave functions of standard relativistic quantum theory are cross sections of a line bundle over projective ray space with an action of the group $SL(2, C)$; physically the group elements are interpreted in terms of Lorentz rotations and boosts of observer reference frames (cf., for example, Chap. 1 of Streater and Wightman, 1964). If we are to try out the idea that the mathematical occurrence of $SL(2, C)$ from the inducing construction should be identified with the occurrence in physics of restricted Lorentz invariance, then we are led to the interpretation of each distinct cross section representation ψ of a fundamental quantum ray space trajectory in P_1C as somehow corresponding to a distinct observer viewpoint, or more generally, a distinct observer preparation, of the same fundamental system. A transformed cross section $U_g\psi$ would then give the appropriate wave function to represent the system in an observer frame or lab differing from the original by the Lorentz transformation corresponding to $g \in SL(2, C)$. If this interpretation can be consistently carried through, several puzzling features of the standard formulation of quantum theory can be explained. First, foundational studies have never revealed a fully satisfactory explanation for the assumed necessity of infinite-dimensionality in the fundamental space of quantum theory. In the proposed interpretation we see that there is no need to postulate infinite-dimensionality, since over even the lowest-dimensional projective spaces there are available natural infinite-dimensional cross-section spaces to handle any structures that cannot survive in lower dimensions. Second, the association of frame-dependent quantities with the "upstairs" space rather than with the base manifold of quantum systems seems to fit with the fact that the quantities standardly assigned unbounded operator representations (e.g., position, momentum), and so requiring infinite dimensions, do not characterize intrinsic properties of the system under observation, but rather properties of the system relative to some observing frame. Thus we see the possibility of a natural distinction between physical observables intrinsic to a system and those that are more in the nature of parameters labeling different ways of representing the system, essentially involving many other systems of a complicated macroscopic or microscopic type. This would bring a concomitant clarification of the mysterious role of classical continuous spectrum observables in modern quantum field theory and would provide also the proper setting for field quantities satisfying boson commutation relations and representing interactions of systems—quantities that even classically require an infinity of degrees of freedom. This line of thought points back to the previously mentioned fact that quantities satisfying commutation relations requiring infinite-dimensionality correspond to a particular coordinatization of projective space rather than to its intrinsic structure.

However this may ultimately turn out, if we simply assume for the time being that the classical and quantum physics of relativistic electrodynamics finds its proper mathematical milieu in the cross-section structure of bundles over the simplest complex projective submanifolds of projective N-space (P_1C), then we can go on to note that the $n = 2$ case of the inducing construction gives the mathematics standardly associated with strong interaction physics—irreducible unitary representations of $SL(3, C)$ with reducible representations of $SU(3)$. This suggests, of course, the possibility that the appearance of hadrons in physics is simply the first encounter with systems and processes that intrinsically require two projective space dimensions for their proper mathematical description, and hence three dimensions of the ambient Hilbert space, with correspondingly richer cross section and bundle structure available over the more complicated trajectory structure possible in the base manifold; it also suggests the association of increasing values of the parameter n in the inducing construction with the quantized differences in interaction types uncovered in modern physics, as well as with the increasing orders of magnitude of the energies necessary to uncover and study the fundamental particles and processes involved in the various interaction types.

Now we will investigate in some detail below the possibility that the well-known isomorphism between Minkowski space and the two-by-two self-adjoint matrices representing quantum theoretical observables over P_1C (ambient space C^2) is not mere coincidence, but rather the way that the observable quantities on the simplest possible processes in quantum ray space should manifest themselves. If this interpretation is accepted, then the four-dimensional appearance of the macroscopic universe would be uniquely associated with electromagnetic means of observation (proper to the $n = 1$ case of the inducing construction), and standard relativistic physics should give way to the appropriate analog—the nine-dimensional space of self-adjoint operators on C^3 (the ambient space for P_2C) when the strong interaction predominates. Is there evidence for this breakdown of the four-dimensional structure of the observable universe in modern physics? At the fundamental particle level it is hard to say, since the normal operational definitions of the standard space-time variables seem to become fuzzy, but it is interesting to note that, at the cosmological level, the standardly accepted sequence of final states of stars of increasing mass (white dwarfs, neutron stars, black holes) requires the introduction of strong interaction physics for any detailed understanding of the type of matter that forms the gateway to the infinite mass density predictions of general relativity. If we apply the strict scientific logic that has prevailed in the past when a classical theory has predicted actual infinities (ultraviolet and infrared catastrophes, for example), we should interpret the singularities of general relativity as the first signs of limitations of applicability of the theory. The mathematical model proposed in the present paper provides a framework for understanding why the detection in cosmological physics of hadron matter and beyond ($n \geq 2$) should be associated with difficulties in applying a theory based on the four-dimensional structures proper to electrodynamics; more positively, the model predicts definite higher-dimensional structures to replace and generalize the lower-dimensional ones as needed. We omit here the obvious further conjectures concerning the role of higher orders [$n = 3$, $SL(4, C)$, $SU(4)$, etc.], except to note that the mathematical direction indicated by the inducing construction seems to fit the pattern of current theoretical research into the unification of the fundamental interactions of physics.

The interpretation of physical theory proposed here depends on being able to identify the Lorentz invariance found in nature with the natural action of $SL(2, C)$ on the cross sections of the line bundle over P_1C given by the inducing construction; P_1C is understood as the complex ray space generated by the simplest possible nonstationary evolution of a quantum system, and cross sections are given their standard interpretation as wave functions. Let us look at some of the details of the inducing construction in this fundamental case. [For background and further details, see Hermann (1966) and Mackey (1968).] In general the construction depends on the fact that P_nC is a homogeneous space of the group $SL(n+1, C)$, i.e., that P_nC can be realized as a left coset space $SL(n+1, C)/G_0 = \{gG_0; g \in SL(n+1, C)\}$, where G_0 is a closed subgroup of $SL(n+1, C)$. Then the Hilbert space of cross sections of a line bundle over P_nC can be represented either directly as complex functions ψ on P_nC, square integrable with respect to the class of measures associated with the Riemannian metric on projective space, or, equivalently and for many purposes more conveniently, as the space of functions $\tilde{\psi}$ on $SL(n+1, C)$, square integrable with respect to the natural Haar measure, and having the property

$$\tilde{\psi}(gh) = \sigma(h^{-1})\tilde{\psi}(g), \qquad g \in SL(n+1, C), \qquad h \in G_0$$

where σ is a representation of G_0 in C. To each such representation of G_0 there corresponds a complex line bundle E_σ over P_nC such that the Hilbert space generated by the square-integrable cross sections sustains a unitary representation of $SL(n+1, C)$. In relation to P_nC, of course, G_0 is simply the isotropy subgroup of some fixed projection under the natural action of $SL(n+1, C)$ on the ambient space C^{n+1}. The two different modes of presentation of cross sections—as functions ψ over P_nC or as functions $\tilde{\psi}$ over $SL(n+1, C)$ satisfying sufficient requirements to define functions on P_nC—are equivalent in the sense that given one we can uniquely construct the other and vice versa, and all relevant bundle structures translate properly. We will consistently use the tilde ($\tilde{}$) to distinguish the various structures in their presentation over $SL(n+1,C)$. Note that the cross sections referred to in our original statement of the results of the inducing construction are in the presentation over the group, and the tensorial character of the representations results precisely from the imposition of the necessary conditions to make everything work over $SL(n+1,C)$. Full details are contained in Hermann (1966), but briefly, the cross sections are given by

$$\tilde{\psi}_A(g)=\frac{A^{j_1\cdots j_k}_{i_1\cdots i_{k+r}}z_{i_1}(g)\cdots z_{i_{k+r}}(g)\bar{z}_{j_1}(g)\cdots\bar{z}_{j_k}(g)}{|z(g)|^{2k+r}}$$

where $z(g)\in C^{n+1}$ is the first column in an initially chosen matrix realization of $SL(n+1,C)$.

Specializing to $n=1$ and using the Dirac-style notation $\hat{\psi}=|\psi\rangle\langle\psi|$ for elements of P_1C, where $|\psi\rangle$ represents a corresponding unit vector in some "abstract" ambient space, we will think of ψ, $\tilde{\psi}$, etc. (without the ket symbol $|\rangle$), as representing some sort of "functional" realization of the abstract vectors, to be specified more precisely in context. Concretely realizing and parametrizing P_1C as the set of matrices of the form

$$\hat{\psi}(\theta,\varphi)=\begin{pmatrix}\cos^2\theta & \cos\theta\sin\theta\, e^{-i\varphi}\\ \cos\theta\sin\theta\, e^{i\varphi} & \sin^2\theta\end{pmatrix}$$

simultaneously realizes the action of

$$g=\begin{pmatrix}z_1 & z_3\\ z_2 & z_4\end{pmatrix}\in G=SL(2,C),\qquad z_1z_4-z_2z_3=1$$

on P_1C by the formula

$$g(\hat{\psi})=\frac{g\hat{\psi}g^+}{\mathrm{tr}(g\hat{\psi}g^+)}=\frac{g\hat{\psi}g^+}{\langle\psi|g^+g|\psi\rangle}$$

The characterization of P_1C above, of course, results directly from imposing self-adjoint, idempotent, trace 1 conditions on 2×2 matrices. We note that distinct geodesics connecting the orthogonal pair of projections $\hat{\psi}_0=\hat{\psi}(0,\varphi)$, $\hat{\psi}_1=\hat{\psi}(\pi/2,\varphi)$, are labeled by distinct fixed values of the parameter $\varphi\in[0,2\pi)$. The matrix representation used is that defined by a basis $|\psi_0\rangle,|\psi_1\rangle$ corresponding to $\hat{\psi}_0,\hat{\psi}_1$, and in terms of this basis the general unit vector corresponding to $\hat{\psi}(\theta, \varphi)$ is

$$|\psi_{\theta,\varphi}\rangle = e^{i\xi}\big(\cos\theta\,|\psi_0\rangle + \sin\theta\, e^{i\varphi}|\psi_1\rangle\big)$$

Choosing $\xi = 0$ and incorporating $e^{i\varphi}$ into the choice of basis vector $|\varphi_1\rangle$ provides a real representation for each geodesic and makes evident the isomorphism of geodesics with both P_1R and the circle.

The isotropy subgroup $G_0 \subset G$ leaving $\hat{\psi}_0 = (\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix})$ invariant consists of all matrices of the form $h = (\begin{smallmatrix}w_1 & w_3 \\ 0 & w_1^{-1}\end{smallmatrix})$, and so is a four-parameter subgroup of the six-parameter $G = SL(2, C)$. The representations of G_0 in the complex numbers are of the form

$$\sigma_r(h) = (w_1/|w_1|)^r$$

and it is to each of these (with r an integer) that a line bundle over P_1C and G can be made to correspond. The assignment of projections to group elements is made in the obvious way by defining, for $g = (\begin{smallmatrix}z_1 & z_3\\ z_2 & z_4\end{smallmatrix})$,

$$\hat{\psi}(g) = g(\hat{\psi}_0) = \frac{1}{1 + |z_2/z_1|^2}\begin{pmatrix} 1 & \bar{z}_2/\bar{z}_1 \\ z_2/z_1 & |z_2/z_1|^2 \end{pmatrix}$$

i.e., we simply use the natural action of the matrices g on the projection $\hat{\psi}_0$, and we see that $\zeta(g) = z_2/z_1 = \rho e^{i\varphi}$ provides coordinates (ρ, φ) for P_1C. Comparing with our earlier matrix realization of projections, we have $\rho = \tan\theta$. Choosing $r = 0$ for simplicity, so that σ_0 is just the identity representation $\sigma_0(h) = 1$, $h \in G_0$, we see that the only condition needed on $\tilde{\psi}$: $G \to C$ in order to define a cross section is that $\tilde{\psi}$ should be homogeneous of degree zero in the group variables z_1, z_2. Writing $\hat{g} = g(\hat{\psi}_0)$, we have as the relation between the two different modes of presentation of cross sections

$$\tilde{\psi}(g) = \tilde{\psi}(z_2/z_1) = \psi(\hat{g})$$

and corresponding infinite-dimensional representations of the group G are specified in both presentations by

$$[R_{g_1}\psi](\hat{g}) = \psi[g_1^{-1}(\hat{g})]$$

$$[\tilde{R}_{g_1}\tilde{\psi}](g) = \tilde{\psi}[g_1^{-1}g]$$

Unitarity is then achieved by defining

$$[U_{g_1}\psi](\hat{g}) = \left[J_{g_1,g}^{1/2+is} R_{g_1}\psi\right](\hat{g})$$

where s can be an arbitrary real parameter and $J_{g_1,g} = J_{g_1^{-1},g}$ is the Jacobian of the coordinate transformation induced by g_1, with of course the corresponding definition for $\tilde{U}$ in terms of $\tilde{J}$ over group space. The inclusion of the proper power of the Jacobian, of course, simply serves to cancel out the Jacobian in the integral $|\psi|^2$ that defines the inner product for cross sections. We can now assign to any individual projection $\hat{g} = g(\hat{\psi}_0)$ the wave function $\tilde{\psi}_{\hat{g}}$: $G \to C$: $g_1 \to \tilde{\psi}_{\hat{g}}(g_1) = [\tilde{U}_{g_1}\tilde{\psi}](g)$. Thus we have the full machinery for formulating cross sections either over projective space or over a fundamental symmetry group of physics. The idea that quantum theory has its most natural setting when formulated in terms of the functions, differential equations, and generalized Fourier analysis available over groups is not new, and has been investigated in a nonrelativistic Abelian group context by J. Malzan (1974) in an interesting paper "Quantum Mechanics Presented as Harmonic Analysis." We intend to investigate this line of thought thoroughly, but at present we can only offer the conjecture that the interpretation given by Malzan to the fundamental frame-dependent observables of physics (momentum, energy, angular momentum, position, etc.) in terms of analysis over groups should prove a fruitful tool in interpreting the corresponding observables in the intertwined structure of wave functions over projective space and wave functions over groups provided by the inducing construction.

Returning now from our sketch of the mathematical details to the problems of physical interpretation, and short of a full answer, what tentative conclusions might be conjectured? If the association of our freedom to choose a variety of different cross section representations of the same basic quantum trajectory with the freedom to subject the same fundamental physical system to a variety of different circumstances (different system preparations, different macroscopic observing techniques, different fields, etc., analogous to different boundary conditions) is accepted, and the elements of $SL(2,C)$ have their standard interpretation as Lorentz frame changes, then these two observer "degrees of freedom" should be able to account for the full array of quantum electrodynamical phenomena associated with a charged spin-1/2 particle. A clue that this is moving in the right direction is provided by the fact that the standard treatment of such a particle in a constant magnetic field results in a geodesic trajectory $\hat{\psi}(\theta)$ for the system represented in ray space, with the parameter interpreted as precessional angle about an axis defined by the magnetic field. Allowing accelerated observing frames and various initial field strengths already account for a great variety of phenomena, and the fact that the frequencies associated with such systems provide one of the best operational means of defining precise fundamental time standards gives some weight to the possibility that ultimately our whole macroscopic four-dimensional electromagnetically conditioned worldview may be built up as a mosaic of interactions with just such elementary systems. This line of thought is reminiscent of an old but durable idea in quantum physics: the apparently complete interchangeability and indistinguishability of particles of the same type at the microscopic level, say electrons, should receive a fully satisfactory explanation only in terms of a theoretical structure that predicts in some sense just one underlying "electronic field," of which we, with our various observing means, catch only facets and glimpses. The inducing construction, with its hierarchy of levels, each with an immense richness of structure over it, seems to provide a step towards making such a model precise.

A seemingly peripheral issue deserves mention, since it gives indications of becoming important. If we start with real instead of complex projective space and look at the Hamiltonian dynamics possible over a geodesic, P_1R, using the standard techniques of modern analytical dynamics (Abraham and Marsden, 1978) we find that we must work in a four-dimensional manifold (the tangent bundle of the cotangent bundle) over the original one-dimensional space. The metric defined by dynamics for the four-dimensional space has signature $(+,+,+,-)$, with the last $(+,-)$ part corresponding to the symplectic metric of the cotangent bundle. When we use the natural complexification of the cotangent bundle $[(q,p) \to q+ip]$ and translate everything over to P_1C, we seem to be led inevitably to the group $SL(2,C)$ as an appropriate means of defining symplectomorphisms (canonical transformations) of the dynamical structure. If we can give this a satisfactory physical interpretation, we would simultaneously have an explanation for the complex numbers in standard quantum theory (the complexification natural to Hamiltonian dynamics over a cotangent bundle) and for the fact that the complex projective space of quantum theory seems to correspond to classical phase space (coordinate and momentum space).

Finally, we come to a "coincidence" that the author finds hard to ignore: the well-known isomorphism between Minkowski space and the self-adjoint operators on C^2, the ambient Hilbert space for P_1C. If our basic supposition is correct, that the fundamentality of spin $1/2$ in physics comes about from the fundamentality of geodesics in projective space, with corresponding geometric confinement of the evolution of such a system to a two-dimensional subspace of the ambient Hilbert space, then the self-adjoint operators on 2-space (observables on P_1C systems) should have correspondingly fundamental significance. This stands out most clearly when we note that arbitrary observables A on a general ambient Hilbert space for quantum theory compress down to 2×2 self-adjoint operators X_A when applied to systems $\hat{\psi}(\theta, \varphi)$ confined to P_1C, i.e., if P is the two-dimensional projection corresponding to the subspace generated by $\hat{\psi}(\theta, \varphi)$, then the trace formula for computing quantum expectation values gives

$$\bar{A} = \mathrm{tr}[A\hat{\psi}(\theta, \varphi)] = \mathrm{tr}[PAP\hat{\psi}(\theta, \varphi)] = \mathrm{tr}[X_A\hat{\psi}(\theta, \varphi)] = \bar{X}_A$$

Thus the mapping $A \to X_A = PAP$ serves as a projection of general observables down to 2×2 observables, and we see that all possible quantum theoretical information that can be determined on a system confined to P_1C comes via the mediation of 2×2 self-adjoint operators.

Now the significance of the space $\mathfrak{X}$ of 2×2 self-adjoint operators has previously been recognized in quantum theory in terms of bases $\sigma_\mu \in \mathfrak{X}$, $\mu = 0, 1, 2, 3$, containing the Pauli operators interpreted as spin observables, with $\sigma_0 = P = I_{C^2}$. Penrose (1971) has suggested a possibly more fundamental significance in a combinatorial approach to building up space itself from the quantum rules the spin observables obey (spin networks). A still more fundamental significance might be hidden in the supposed coincidence referred to earlier if a direct interpretation of the space of observables $\mathfrak{X}$ as either energy-momentum or time-position Minkowski space could be found —something along the lines of interpreting the trace formula $\mathrm{tr}(X\hat{\psi})$ as the expectation value that system $\hat{\psi}$ would be detected at (or by observing system corresponding to) Minkowski space point X. The simplest tensorial cross sections given by the inducing construction are of the form $\mathrm{tr}(X\hat{\psi}) = \langle\psi|X|\psi\rangle$, but the full resolution of this line of investigation awaits further work, because of difficulties of interpretation involving probabilities, probability amplitudes, etc. However, let us look at some leads that might motivate such an investigation. The isomorphism between Minkowski space and the self-adjoint operators $X\in\mathfrak{X}$ is specified by

$$(x_0, x_1, x_2, x_3) \to \begin{pmatrix} x_0 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 - x_1 \end{pmatrix} = X$$

with the Euclidean inner product $\langle X, Y\rangle$ on $\mathfrak{X}$ transformed to the Minkowski scalar product $X\cdot Y$ by replacing the standard adjoint X^+ in the formula $\langle X, Y\rangle = \frac{1}{2}\mathrm{tr}(X^+ Y)$ by the "classical" adjoint $X^{[+]}$, to give $X\cdot Y = \frac{1}{2}\mathrm{tr}(X^{[+]}Y)$. The classical adjoint

$$X^{[+]} = \begin{pmatrix} x_0 - x_1 & -x_2 - ix_3 \\ -x_2 + ix_3 & x_0 + x_1 \end{pmatrix}$$

is related to the inverse, in the sense that it is the inverse short of division by the determinant. Thus, for $\mathrm{Det}(X) \neq 0$, we have

$$X^{[+]} = \mathrm{Det}(X) X^{-1}$$

and we see that the light cone in Minkowski space corresponds to the noninvertible operators [$\mathrm{Det}(X) = 0$], and these are exactly all multiples of one-dimensional projections, i.e., quantum pure states. The forward light cone and interior correspond to positive operators, with the interior made up of multiples of mixed states. Thus the Minkowski metric locates and classifies quantum states in a way that is suggestive of what one might want if some interpretation is to be given of states as quantum observing events building up our (3+1)-worldview. Obviously, much further work in the interpretation of quantum theory must be done before anything can be made of this suggestion, but again it seems to be pointing toward geodesics of projective space with their associated two-dimensional and four-dimensional structures as fundamental roots of relativity in physics. In fact the structures provided by the inducing construction seem to be so intrinsically relativistic that one of the problems encountered in interpreting the formalism so far has been the difficulty of finding simple nonrelativistic test case examples (Schrödinger equation solutions, etc.) for guidance. From the proper perspective, this can be viewed optimistically, however, since we should probably look with suspicion on any model that allows easy and direct formulation of nonrelativistic results except as low-energy limits. Thus we can hope that, when fully worked out, the results of the inducing construction will satisfy the criterion, attributed to Einstein, for mathematical methods in physics: they should "make the good easy and the bad difficult."

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